Mathematics for Machine Learning: Homework 2
Deadline is 30.07.2020

July 21, 2020

1. Calculate determinants of the following matrices.

   a) $\begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$,

   b) $\begin{bmatrix} 2 & 5 & 1 & 3 \\ 4 & 1 & 7 & 9 \\ 6 & 8 & 3 & 2 \\ 7 & 8 & 1 & 4 \end{bmatrix}$.

2. Consider linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbb{R}^n$ and

$$\begin{array}{rcrcrcr} \mathbf{x}_1 & = & 2\mathbf{b}_1 & + & 3\mathbf{b}_2 & - & \mathbf{b}_3 \\ \mathbf{x}_2 & = & \mathbf{b}_1 & - & 2\mathbf{b}_2 & + & 2\mathbf{b}_3 \\ \mathbf{x}_3 & = & \mathbf{b}_1 & - & 9\mathbf{b}_2 & + & a\mathbf{b}_3 \end{array}$$

   For which value of the parameter a are the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^n$ linearly dependent?

3. Prove that the following sets are linearly independent. Are those bases for the corresponding spaces?

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^4,\ \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3, \mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

4. Determine whether there is a subset of vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}.$$

   which is a basis for $\mathbb{R}^4$.

5. Find the rank of the following matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \\ 2 & 5 & -3 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \\ 3 & 9 \end{bmatrix}.$$

6. Show that the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$, is an automorphism.

7. Determine the coordinate vector of $\mathbf{x}$ with respect to the ordered basis B for $\mathbb{R}^4$, where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

8. Find the Manhattan norm (ℓ_1 norm), the Euclidean norm (ℓ_2 norm) of the following vectors:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 12 \\ -3 \\ 4 \end{bmatrix}.$$