Mathematics for Machine Learning: Homework 5
Deadline is 20.08.2020

August 13, 2020

1. Using the definition of limit, prove that

   a) $\lim\limits_{x\to 2} \dfrac{x^2+4x-5}{x^2-1} = \dfrac{7}{3}$, b) $\lim\limits_{x\to+\infty} \dfrac{3x^2+5x-7}{x^2-1} = 3$.

2. Calculate the limit

   a) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{\sin^3 x}$,

   b) $\lim\limits_{x\to+\infty} \left(\dfrac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}}$,

   c) $\lim\limits_{x\to 0} \dfrac{\sqrt[5]{1+\sin x}-1}{\ln(1+\operatorname{tg} x)}$,

   d) $\lim\limits_{n\to+\infty} \left(\dfrac{\sqrt[n]{2}+\sqrt[n]{3}}{2}\right)^n$.

3. For which values of a the following function will be continuous on $\mathbb{R}$

$$f(x) = \begin{cases} \sin|x| - \ln|x|, & \text{if } |x| \geq 1 \\ ax^2 - 1, & \text{if } |x| < 1. \end{cases}$$

4. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there exists $c \in [0,1]$ such that $f(c) = c$.

5. Let $f(x) = x(x-1)(x-2)(x-3)(x-4)$. Prove that the all roots of the equation $f'(x) = 0$ are lying inside the interval $(0,4)$.

6. Find all extremum points of f

   a) $f(x) = 2x^2 - x^4$,

   b) $f(x) = xe^{-x}$,

   c) $f(x) = \cos x + \frac{1}{2}\cos 2x$,

   d) $f(x) = e^x \sin x$.

7. Find maximum and minimum values of f in the given interval

   a) $f(x) = 2^x$, $x \in [-1,5]$,

   b) $f(x) = x^3 - 3x^2 + 6x - 2$, $x \in [-4,3]$,

   c) $f(x) = \sqrt{x}\ln x$, $x \in (0,1]$,

   d) $f(x) = |x| + \dfrac{x^3}{3}$, $x \in [-1,1]$.

8. Calculate the integral

a) $\displaystyle\int \frac{dx}{x\ln x}$,

b) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1}}$,

c) $\displaystyle\int \frac{dx}{x^2-x}$,

d) $\displaystyle\int \frac{dx}{\cos x}$.

9. Investigate convergence of the series

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+1}{n+3} \sin\frac{1}{n^2+2}$,

b) $\displaystyle\sum_{n=1}^{\infty} \left(3^{\frac{1}{n}}-1\right) \sin\frac{\pi}{n}$,

c) $\displaystyle\sum_{n=1}^{\infty} 2^n \left(\frac{n}{n+1}\right)^{n^2}$,

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3\, 4^{3n}}$.